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Thermally developing Poiseuille flow with a non-uniform entrance temperature when the viscous heat generation is significant

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Abstract

The classical problem of thermally developing Poiseuille flow in the presence of viscous dissipation (the 'Graetz–Brinkman problem') is reviewed in this paper. Taking into account that the traditional assumptions of a uniform entrance temperature and of Poiseuille velocity profile are mutually exclusive concepts when the frictional heat generation is significant, in the present paper a non-uniform entrance temperature profile is considered. This non-uniform 'initial condition' is not prescribed, but it is deduced from the energy balance equation in a consistent way as the fully developed temperature profile of the Poiseuille flow under isothermal boundary conditions. Both for the dependence of the local Nusselt number on the Brinkman number and the developing temperature field, substantial differences have been found compared with the traditional case. The consequences of this sensitive dependence on the initial condition are discussed in detail.

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Nomenclature

- Br Brinkman number, dimensionless, equation (12)
- d_h hydraulic diameter (m), $d_\mathrm{h} = 2r_0$
- D_n dimensionless coefficients, equation (28)
- c_p specific heat at constant pressure($\mathrm{J\,kg^{-1}\,K^{-1}}$), equation (2)
- H_n dimensionless coefficients, equation (29)
- k thermal conductivity ($\mathrm{W\,m^{-1}\,K^{-1}}$), equation (2)
- M confluent hypergeometric function, equation (21)

Nu	Nusselt number, dimensionless, equation (35)
N_n	dimensionless normalization constant, equation (24)
p	pressure, ($\mathrm{N\,m^{-2}}$), equation (1b)
Pe	Peclét number, $Pe = u_m d_h/\alpha$, dimensionless,
q_w	wall heat flux ($\mathrm{W\,m^{-2}}$), equation (36)
r	radial coordinate (m)
r_0	duct radius (m)
R_n	dimensionless eigenfunctions, equation (19)
S	series, equations (41)
T	temperature (K)
T_b	bulk temperature (K), equation (37)
T_*	temperature scale of viscous dissipation (K), equation (3b)
u	velocity ($\mathrm{m\,s^{-1}}$), equation (1)
u_m	average velocity ($\mathrm{m\,s^{-1}}$), equation (1)
z	axial coordinate (m)

Greek symbols

α	thermal diffusivity ($\mathrm{m^2\,s^{-1}}$), $\alpha = k/(\rho c_p)$
δ_{nm}	Kronecker symbol, equation (24)
λ_n^2	dimensionless eigenvalues, equation (19)
μ	dynamic viscosity ($\mathrm{kg\,m^{-1}\,s^{-1}}$), equation (2)
ρ	fluid density, ($\mathrm{kg\,m^{-3}}$)
η	dimensionless radial coordinate, equation (8a)
ζ	dimensionless axial coordinate, equation (8b)
Θ	dimensionless temperature, equation (8c)

Subscripts

as	asymptotic
e, en	entrance condition
n	summation index

Superscripts

$\sim$	dimensionless quantity, equations (18), (42), (43)

1. Introduction

The Graetz problem [1] and its numerous extensions are of basic importance for convective heat transfer. All of them concern the steady thermal development of a hydrodynamically fully developed forced convection duct flow as a consequence of an abrupt change of the thermal boundary conditions at some axial station, referred to usually as the 'thermal entrance station' of the flow.

The classical Graetz problem of the Poiseuille flow for *isothermal-to-isothermal* wall conditions (i.e. for a sudden jump of the wall temperature at the thermal entrance station, from a constant to another constant value) has been rediscovered and solved independently by Nusselt [2]. Since these pioneering papers, the thermal entrance problem has produced an enormous scientific literature. The bulk of the results has been summarized in the comprehensive overviews by Shah and London [3] and Kakac and Yener [4].

One of the major extensions of the Graetz problem concerns the effect of the internal heat generation by viscous dissipation. The importance of this effect for a thermally developing flow in narrow ducts (capillaries) has first been emphasized by Brinkman [5]. On this reason, a Graetz problem in the presence of viscous dissipation is called usually a Graetz–Brinkman problem. First of all, it is worth mentioning here that the effect of viscous dissipation is significant not only in capillary flows but in all duct flows of high-Prandtl number fluids (e.g. engine oils, crude oils, etc), for large and even moderate values of the Peclét number. The mathematical analysis of the Graetz–Brinkman problem is simplified in such cases by the fact that the heat transport by axial conduction is negligible in comparison with the heat released by viscous dissipation and transported by the moving fluid [5]. However, in the case of low-Prandtl number fluids (e.g. liquid metals), for small and even moderate values of the Peclét number, the opposite is true: compared with the heat generation by internal friction, the axial conduction becomes the dominant effect. The latter effect on the thermally developing flow for *isothermal-to-isothermal* boundary conditions has been investigated with the aid of the traditional method of the Graetz problems (the separation of variables) by Lahjomri and Oubarra, [6]. The cases of real technical interest in which both the effects of viscous dissipation and of axial heat conduction would become simultaneously significant, are quite rare.

Subsequent to Brinkman's paper [5] concerned with the *isothermal-to-isothermal* and the *isothermal-to-adiabatic* boundary conditions, considerable analytical and numerical research works have been directed on different extensions and applications of the Graetz–Brinkman problem (see, e.g., Toor [7], Gill [8], Ou and Cheng [9, 10] and references therein). The analytical solution of the *isothermal-to-isothermal* as well as of the *isothermal-to-isoflux* Graetz–Brinkman problem has first been given by Ou and Cheng [9, 10] and later also by Basu and Roy [11]. The case of the Poiseuille flow with *isothermal-to-convective* boundary conditions has been considered by Lin *et al* [12] and the Graetz–Brinkman problem for the slug flow with *isothermal-to-axially variable heat flux* boundary conditions has more recently been solved by Barletta and Zanchini [13]. The novel developments concerning the Graetz–Brinkman problem of power law fluids have been reviewed by Valkó [14]. Valkó's paper [14] also reports a new and efficient solution technique based on the Galerkin method in combination with the Laplace transformation, which allows for the most general linear boundary conditions in the range of the thermally developing flow.

A joint feature of the most previous investigations of the Graetz–Brinkman problem is the assumption of a uniform entrance temperature profile, $T_{\mathrm{en}} = \text{constant}$. The present work departs from all these studies by relaxing precisely this basic assumption in favour of a non-uniform entrance condition which is obtained in a natural way as the fully developed temperature profile of a Poiseuille flow under isothermal boundary conditions when viscous dissipation is significant. (This fully developed temperature profile can be prepared by starting the flow 'infinitely far' upstream from the thermal entrance station $z = 0$.) There are two basic arguments to renounce on the traditional uniform entrance condition, namely:

1. In the presence of viscous dissipation, $T = \text{constant}$ is not a solution of the thermal energy equation associated with the Poiseuille flow. Thus the assumption $T_{\mathrm{en}} = \text{constant}$ violates in fact the first principle of thermodynamics in the upstream section of the duct.
2. The non-uniform entrance temperature profile ($T_{\mathrm{en}} \neq \text{constant}$) which solves the mentioned energy equation along with the isothermal boundary condition, leads to a dependence of the local Nusselt number (see equation (35)) on the Brinkman number (see equation (12)) which deviates from that corresponding to $T_{\mathrm{en}} = \text{constant}$ (see figures 1(*a*) and (*b*)) substantially.

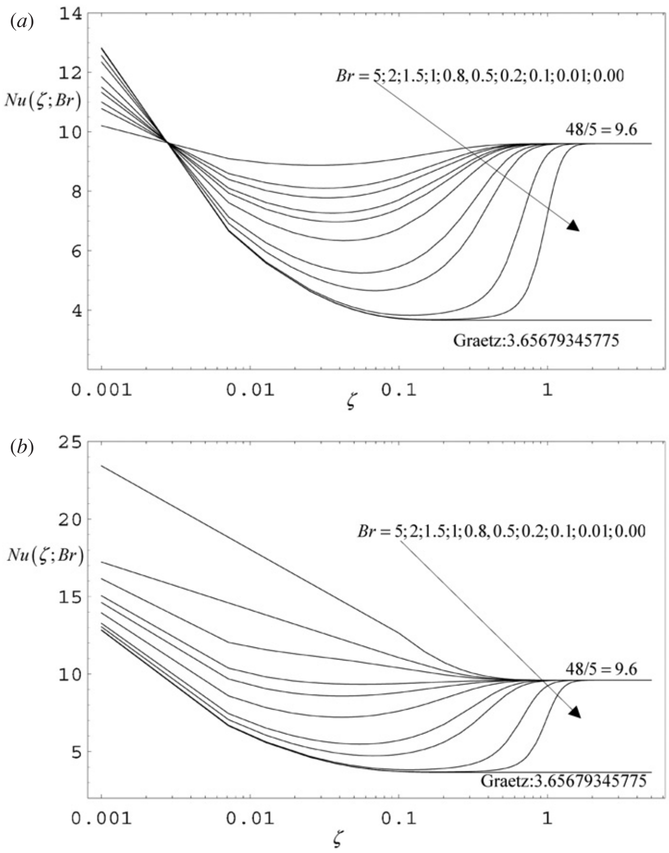

Figure 1. (*a*) The local Nusselt number corresponding to the non-uniform entrance temperature plotted as a function of ζ for different positive values of Br. (*b*) The local Nusselt number corresponding to the uniform entrance temperature plotted as a function of ζ for different positive values of Br.

Accordingly, the main issue of the present paper is a detailed investigation of the effect of the non-uniform entrance temperature mentioned above on the heat transfer characteristics of the thermally developing Poiseuille flow when the viscous dissipation is significant.

It is also worth mentioning here that the above aspect (i) has already been recognized by Gill [8]; however, strangely enough, his early insight has not penetrated the later research literature.

2. Model and governing equations

We consider the Poiseuille flow in a circular duct of radius r_0. The fluid velocity as a function of the radial coordinate r is given by

$$u(r) = 2u_\mathrm{m}[1 - (r/r_0)^2], \tag{1a}$$

$$u_\mathrm{m} = \frac{r_0^2}{8\mu}\left(-\frac{\mathrm{d}p}{\mathrm{d}z}\right), \tag{1b}$$

where u_m denotes the average velocity. The z axis of the coordinate system coincides with the cylinder axis.

In the upstream range $z < 0$, the duct wall is held at the constant temperature T_e. At the entrance station $z = 0$ of the thermally developing flow regime, the wall temperature changes abruptly from T_e to another constant value $T_w \neq T_e$ which is kept constant in the whole downstream range $z > 0$. Both the so-called 'fluid cooling' ($T_w < T_e$) and the $T_w > T_e$ 'fluid heating' ($T_w > T_e$) situations are of physical and engineering interest (see section 4). We further assume that the axial heat conduction is negligible, while the viscous dissipation is significant. The physical properties are also considered constant. Under these conditions the temperature field $T(r, z)$ of our forced convection flow is governed by the thermal energy equation

$$\rho c_p u \frac{\partial T}{\partial z} = \frac{k}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \mu \left(\frac{\mathrm{d}u}{\mathrm{d}r} \right)^2, \qquad 0 < r < r_0. \tag{2}$$

The last term on the right-hand side of equation (2) describes the contribution of the heat released by viscous dissipation to the overall heat balance of the flow.

It is clearly seen that, in the presence of viscous dissipation, the traditional assumption of a uniform entrance temperature violates the energy equation (2). Hence, the entrance temperature which is consistent with the first principle of thermodynamics must be a non-uniform solution of equation (2) satisfying the boundary condition $T|_{r=r_0} = T_e$ for $z \leqslant 0$. It is easy to show that temperature profile, $T = T_{en}(r)$,

$$T_{en}(r) = T_e + T_*[1 - (r/r_0)^4], \tag{3a}$$

$$T_* = \frac{\mu u_m^2}{k}, \tag{3b}$$

of the thermally fully developed Poiseuille flow satisfies this requirement exactly. (As mentioned above, the fully developed temperature profile can be prepared by starting the flow 'infinitely far' upstream from the thermal entrance station $z = 0$.) This circumstance yields the main motivation for the present work, which removes the traditional uniform entrance condition $T_{en} =$ constant in favour of the non-uniform one given by equation (3).

In the downstream range $z > 0$, the Poiseuille flow (1) is still maintained, but it becomes thermally developing under the isothermal boundary conditions

$$T(r_0, z) = T_w \neq T_e, \qquad z > 0. \tag{4}$$

Having in mind the axial symmetry of equation (2) and of the boundary condition (4), the centreline condition

$$\left. \frac{\partial T}{\partial r} \right|_{r=0} = 0, \qquad z \geqslant 0 \tag{5a}$$

or

$$T|_{r=0} = \text{finite} \qquad \text{for all } z \tag{5b}$$

must also be satisfied (conditions (5a) and (5b) have the same effect on the solution).

Furthermore, due to the parabolic character of equation (2), at the entrance station $z = 0$ of the thermally developing flow also an 'initial condition' is necessary. This initial or thermal entrance condition requires that at $z = 0$ the temperature field $T(r, z)$ coincides with the temperature field (3) of the fully developed entrance flow,

$$T(r, 0) = T_{en}(r) = T_e + T_*[1 - (r/r_0)^4]. \tag{6}$$

In this way, equation (2) along with conditions (4), (5a) and (6) specifies a well-posed problem.

3. Solution

3.1. Asymptotic solution

In the far downstream section of the duct, $z \to \infty$, the flow has lost any memory of the thermal entrance effects which are dominant only in the range of small values of the axial coordinate z. Hence, for $z \to \infty$ the solution of equation (2) becomes independent of z. This solution is in fact the temperature profile of the thermally developed Poiseuille flow in the presence of viscous dissipation similar to the entrance temperature profile (3*a*), but in this case subject to the boundary condition (4), instead of $T|_{r=r_0} = T_e$. Therefore, this asymptotic solution reads

$$T_{as}(r) = T_w + T_*[1 - (r/r_0)^4], \qquad z \to \infty \tag{7}$$

3.2. Nondimensionalization

It is convenient to introduce the dimensionless variables

$$\eta = \frac{r}{r_0}, \tag{8a}$$

$$\zeta = \frac{1}{Pe}\frac{z}{r_0}, \tag{8b}$$

$$\Theta(\eta, \zeta) = \frac{T(r, z) - T_w}{T_e - T_w}, \tag{8c}$$

where Pe is the Peclét number, $Pe = 2r_0 u_m/\alpha$, $\alpha = k/(\rho c_p)$.

With the new variables the energy equation (2), the entrance and the asymptotic temperature profiles (3*a*) and (7) become

$$(1 - \eta^2)\frac{\partial\Theta}{\partial\zeta} = \frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\Theta}{\partial\eta}\right) + 16Br\eta^2, \qquad \zeta > 0, \qquad 0 < \eta < 1 \tag{9}$$

$$\Theta_{en}(\eta) = 1 + Br(1 - \eta^4), \qquad 0 \leqslant \eta \leqslant 1, \qquad \zeta = 0 \tag{10}$$

$$\Theta_{as}(\eta) = Br(1 - \eta^4), \qquad 0 \leqslant \eta \leqslant 1, \qquad \zeta \to \infty, \tag{11}$$

where Br denotes the Brinkman number,

$$Br = \frac{T_*}{T_e - T_w} = \frac{\mu u_m^2}{k\,(T_e - T_w)}. \tag{12}$$

To the cases of 'fluid cooling' ($T_w < T_e$) and 'fluid heating' ($T_w > T_e$), there correspond positive and negative values of the Brinkman number, respectively.

Furthermore, the boundary, the centreline and the initial conditions (4), (5*b*) and (6) go over in

$$\Theta(1, \zeta) = 0, \qquad \zeta > 0 \tag{13}$$

$$\Theta\,(0, \zeta) = \text{finite}, \qquad \zeta \geqslant 0 \tag{14}$$

$$\Theta\,(\eta, 0) = \Theta_{en}(\eta). \tag{15}$$

3.3. The full solution

Equation (9) along with conditions (13)–(15) specifies a well-posed initial and boundary value problem. It is well known that the full solution of this linear problem can be written in the form of the superposition

$$\Theta(\eta, \zeta) = \Theta_{\rm as}(\eta) + \tilde{\Theta}(\eta, \zeta), \qquad (16)$$

where $\tilde{\Theta}(\eta, \zeta)$ satisfies the homogeneous partial differential equation:

$$(1 - \eta^2)\frac{\partial \tilde{\Theta}}{\partial \zeta} = \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial \tilde{\Theta}}{\partial \eta}\right). \qquad (17)$$

The solution procedure in the present case is basically the same as in all classical Graetz-type problems and it consists of the following main steps.

One first represents $\tilde{\Theta}(\eta, \zeta)$ as a linear combination of elementary separable solutions,

$$\tilde{\Theta}(\eta, \zeta) = \sum_{n=1}^{\infty} C_n R_n(\eta)\, {\rm e}^{-\lambda_n^2 \zeta}. \qquad (18)$$

In this way, from equations (17), (18), (13) and (14) the following Sturm–Liouville eigenvalue problem emerges for the radial functions $R_n(\eta)$,

$$\frac{\rm d}{{\rm d}\eta}\left(\eta \frac{{\rm d}R_n}{{\rm d}\eta}\right) + \lambda_n^2 \eta (1 - \eta^2) R_n = 0 \qquad (19)$$

$$R_n(1) = 0 \qquad (20a)$$

and

$$R_n(0) = 1. \qquad (20b)$$

The solution of this problem can be given in terms of Kummer's confluent hypergeometric function $M\,(a, b, x)$ as follows:

$$R_n(\eta) = {\rm e}^{-\frac{1}{2}\lambda_n \eta^2} M\left(\frac{2 - \lambda_n}{4},\ 1,\ \lambda_n \eta^2\right). \qquad (21)$$

The eigenvalues λ_n^2 are obtained from the solutions of the transcendental equation:

$$M\left(\frac{2 - \lambda_n}{4}, 1, \lambda_n\right) = 0, \qquad (22)$$

which is an immediate consequence of the boundary condition (20*a*). On using the properties of the function $M\,(a, b, x)$ (see, e.g., [15]) it can be shown that if λ_n is a solution of equation (22) then $-\lambda_n$ is also a solution. Moreover, $R_n\,(\eta; \lambda_n) = R_n\,(\eta; -\lambda_n)$, such that it is sufficient to consider, for example, the positive roots of equation (22) only.

It is worth emphasizing here that the assumption concerning the entrance temperature profile $\Theta_{\rm en}(\eta)$ does not affect the boundary value problems (19) and (20). The specific form of $\Theta_{\rm en}(\eta)$ only affects the values of the (yet undetermined) coefficients C_n of the linear superposition (18) via the initial condition (15). Indeed, equations (15), (16) and (18) yield

$$\sum_{n=1}^{\infty} C_n R_n(\eta) = \Theta_{\rm en}(\eta) - \Theta_{\rm as}(\eta). \qquad (23)$$

Thus, having in mind that the Sturm–Liouville eigenfunctions associated with different eigenvalues λ_n^2 are orthogonal,

$$\int_0^1 R_n(\eta) R_{\rm m}(\eta) \eta (1 - \eta^2)\, {\rm d}\eta = N_n \delta_{n{\rm m}}, \qquad (24)$$

where the normalization factors N_n can be calculated explicitly, one immediately obtains for the coefficients C_n the expressions

$$C_n = \frac{1}{N_n}\int_0^1 [\Theta_{\text{en}}(\eta) - \Theta_{\text{as}}(\eta)]\, R_n(\eta)\eta(1-\eta^2)\,\mathrm{d}\eta. \tag{25}$$

This expression makes the difference between the usual choice of the uniform entrance temperature profile $T_{\text{en}}(r) = T_{\text{e}}$, i.e. $\Theta_{\text{en}}(\eta) = 1$, and the present fully developed profile given by equation (10) manifest. Indeed, equations (10) and (11) imply

$$\Theta_{\text{en}}(\eta) - \Theta_{\text{as}}(\eta) = \begin{cases} 1 - Br(1-\eta^4), & \text{uniform } T_{\text{en}} \\ 1, & \text{non-uniform } T_{\text{en}}. \end{cases} \tag{26}$$

Therefore, in these two cases the respective coefficients C_n can differ from each other (for non-negligible values of the Brinkman number Br) substantially. In order to make this distinction transparent also in the forthcoming equations, it is convenient to transcribe equation (25), with account on equation (26), into the form

$$C_n = \begin{cases} D_n - Br H_n, & \text{uniform } T_{\text{en}} \\ D_n, & \text{non-uniform } T_{\text{en}}, \end{cases} \tag{27}$$

where

$$D_n = \frac{1}{N_n}\int_0^1 \eta(1-\eta^2)R_n(\eta)\,\mathrm{d}\eta \tag{28}$$

and

$$H_n = \frac{1}{N_n}\int_0^1 \eta(1-\eta^2)(1-\eta^4)R_n(\eta)\,\mathrm{d}\eta. \tag{29}$$

The coefficients D_n are identical to those of the classical Graetz problem. All the coefficients D_n and H_n are independent of the Brinkman number Br. The contribution of H_n to the coefficient C_n arises only when a uniform entrance temperature is assumed but it disappears in the case of the fully developed entrance temperature. The reason is that, in the latter case, the r-dependent terms of expressions (3a) and (7) compensate each other exactly. In other words, in the case of the uniform entrance temperature $T_{\text{en}}(r) = T_{\text{e}}$ the coefficients C_n depend on the Brinkman number Br linearly according to equation (27), while in the case of the non-uniform entrance temperature profile (3a) they are independent of Br and coincide with the Graetz coefficients D_n. Obviously, this circumstance has also further consequences, namely on the local Nusselt number $Nu(z)$ (see below) as well as for dimensionless temperature profile:

$$\Theta(\eta,\zeta) = Br(1-\eta^4) + \sum_{n=1}^{\infty} C_n R_n(\eta)\,\mathrm{e}^{-\lambda_n^2\zeta}, \tag{30}$$

which has been obtained from equations (16), (11) and (18), where the coefficients C_n are given by equation (27).

On using equations (19) and (20), the integral of equation (28) can be calculated explicitly, such that one obtains

$$D_n = -\frac{R_n'(1)}{\lambda_n^2 N_n}, \tag{31}$$

where

$$R_n'(1) = \lambda_n\left(1 - \frac{1}{2}\lambda_n\right)\mathrm{e}^{-\frac{1}{2}\lambda_n} \times M\left(\frac{6-\lambda_n}{4}, 2, \lambda_n\right). \tag{32}$$

The integral (29) can similarly be reduced to the simpler form

$$H_n = \frac{16}{\lambda_n^2 N_n} \int_0^1 \eta^3 R_n(\eta)\,\mathrm{d}\eta. \tag{33}$$

The coefficients D_n and H_n possess the remarkable property that their respective sums equal unity

$$\sum_{n=1}^{\infty} D_n = 1, \tag{34a}$$

$$\sum_{n=1}^{\infty} H_n = 1. \tag{34b}$$

These relationships can easily be obtained by substituting $\eta = 0$ in equation (23) and taking into account equations (20*b*), (26) and (27).

3.4. *The local Nusselt number*

The quantity of the main engineering interest is the local Nusselt number which is defined usually with respect to the bulk temperature $T_\mathrm{b}(z)$ and the hydraulic diameter d_h (in this case $d_\mathrm{h} = 2r_0$) as follows:

$$Nu(z) = \frac{q_\mathrm{w}(z) d_\mathrm{h}}{k\,[T_\mathrm{w} - T_\mathrm{b}(z)]}. \tag{35}$$

Here $q_\mathrm{w}(z)$ denotes the radial wall heat flux,

$$q_\mathrm{w}(z) = k\,\frac{\partial T(r,z)}{\partial r}\bigg|_{r=r_0} \tag{36}$$

and the bulk temperature of the fluid is defined by

$$T_\mathrm{b}(z) = \frac{2}{u_\mathrm{m} r_0^2} \int_0^{r_0} T(r,z) u(r) r\,\mathrm{d}r. \tag{37}$$

After standard calculations one obtains

$$q_\mathrm{w}(z) = \frac{kT_*}{r_0}\left[-4 + \frac{1}{Br}\sum_{n=1}^{\infty} C_n R_n'(1)\,\mathrm{e}^{-\lambda_n^2 \zeta}\right] \tag{38}$$

and

$$T_\mathrm{b}(z) = T_\mathrm{w} + T_*\left[\frac{5}{6} - \frac{4}{Br}\sum_{n=1}^{\infty} \frac{C_n}{\lambda_n^2} R_n'(1)\,\mathrm{e}^{-\lambda_n^2 \zeta}\right] \tag{39}$$

such that

$$Nu(z) = \frac{48Br - 12\sum_{n=1}^{\infty} C_n R_n'(1)\,\mathrm{e}^{-\lambda_n^2 \zeta}}{5Br - 24\sum_{n=1}^{\infty} \frac{C_n}{\lambda_n^2} R_n'(1)\,\mathrm{e}^{-\lambda_n^2 \zeta}}, \tag{40}$$

where the coefficients C_n are given by equation (27).

4. Discussion

4.1. General considerations

For more transparency, it is convenient to introduce the shortcut notations

$$S_1(\zeta) = -\sum_{n=1}^{\infty} D_n R_n'(1)\, \mathrm{e}^{-\lambda_n^2 \zeta}, \tag{41a}$$

$$S_2(\zeta) = -\sum_{n=1}^{\infty} H_n R_n'(1)\, \mathrm{e}^{-\lambda_n^2 \zeta} \tag{41b}$$

$$S_3(\zeta) = -\sum_{n=1}^{\infty} \frac{D_n}{\lambda_n^2} R_n'(1)\, \mathrm{e}^{-\lambda_n^2 \zeta}, \tag{41c}$$

$$S_4(\zeta) = -\sum_{n=1}^{\infty} \frac{H_n}{\lambda_n^2} R_n'(1)\, \mathrm{e}^{-\lambda_n^2 \zeta} \tag{41d}$$

as well as the dimensionless radial heat flux $\tilde{q}_{\mathrm{w}}(\zeta; Br)$ and temperature difference $\Delta\tilde{T}_{\mathrm{b}}(\zeta; Br)$,

$$\tilde{q}_{\mathrm{w}}(\zeta; Br) = \frac{q_{\mathrm{w}}(z)}{(kT_*/r_0)}, \tag{42a}$$

$$\Delta\tilde{T}_{\mathrm{b}}(\zeta; Br) = \frac{T_{\mathrm{w}} - T_{\mathrm{b}}(z)}{T_*}. \tag{42b}$$

With this notation the local Nusselt number (40) is given by

$$Nu(\zeta; Br) = 2\frac{\tilde{q}_{\mathrm{w}}(\zeta; Br)}{\Delta\tilde{T}_{\mathrm{b}}(\zeta; Br)}. \tag{43}$$

The above quantities will be compared for two cases of the Graetz–Brinkman problem, namely:

(i) the case of uniform entrance temperature in the presence of viscous dissipation (as reported by Basu and Roy [11]),

(ii) the case of the fully developed entrance temperature profile (3*a*) in the presence of viscous dissipation (the present paper).

The corresponding expressions of $\tilde{q}_w(\zeta; Br)$, $\Delta\tilde{T}_{\mathrm{b}}(\zeta; Br)$ and $Nu(\zeta; Br)$ are summarized in table 1. One immediately sees that all these quantities are quite different for the uniform and non-uniform initial conditions, respectively. The classical Graetz problem (uniform entrance temperature, viscous dissipation neglected) corresponds to the limiting case $T_* \to 0$, i.e. $Br \to 0$. In this case, equations (38)–(40) yield

$$q_{\mathrm{w}}(z) = -\frac{k(T_{\mathrm{e}} - T_{\mathrm{w}})}{r_0} S_1(\zeta), \tag{44a}$$

$$T_{\mathrm{w}} - T_{\mathrm{b}}(z) = -4(T_{\mathrm{e}} - T_{\mathrm{w}}) S_3(\zeta), \tag{44b}$$

$$Nu(z) = \frac{1}{2}\frac{S_1(\zeta)}{S_3(\zeta)}. \tag{45}$$

Basically, the eigenvalues λ_n^2 and the coefficients D_n, H_n and $R_n'(1)$ occurring in the above expressions can be calculated from equations (22) and (31)–(33) to any desired precision.

Table 1. Thermal characteristics in the cases (i) and (ii) as specified.

	(i) Uniform entrance temperature	(ii) Non-uniform entrance temperature
Dimensionless radial heat flux: $\tilde{q}_{\mathrm{w}}(\zeta; Br)$	$-\left(4+\frac{S_1}{Br}\right)+S_2$, $\tilde{q}_{\mathrm{w}}(\infty; Br)=-4$	$-\left(4+\frac{1}{Br}S_1\right)$, $\tilde{q}_{\mathrm{w}}(\infty; Br)=-4$
Dimensionless temperature difference Wall-bulk: $\Delta\tilde{T}_{\mathrm{b}}(\zeta; Br)$	$-\left(\frac{5}{6}+\frac{4}{Br}S_3\right)+4S_4$, $\Delta\tilde{T}_{\mathrm{b}}(\infty; Br)=-\frac{5}{6}$	$-\left(\frac{5}{6}+\frac{4}{Br}S_3\right)$, $\Delta\tilde{T}_{\mathrm{b}}(\infty; Br)=-\frac{5}{6}$
Local Nusselt number: $Nu(\zeta; Br) = 2\frac{\tilde{q}_w(\zeta;Br)}{\Delta\tilde{T}_b(\zeta;Br)}$	$\frac{(4-S_2)Br+S_1}{\left(\frac{5}{12}-2S_4\right)Br+2S_3}$, $Nu(\infty; Br)=\frac{48}{5}$	$\frac{4Br+S_1}{\frac{5}{12}Br+2S_3}$, $Nu(\infty; Br)=\frac{48}{5}$

Table 2. The first 10 eigenvalues and coefficients calculated to 12 significant digits.

n	λ_n	D_n	H_n
1	2.704 364 419 88	+1.476 435 406 68	+1.353 410 834 21
2	6.679 031 449 35	−0.806 123 895 55	−0.504 027 201 46
3	10.673 379 5381	+0.588 762 153 61	+0.232 180 064 23
4	14.671 078 4627	−0.475 850 426 24	−0.132 358 829 71
5	18.669 871 8645	+0.405 021 810 71	+0.085 381 851 652
6	22.669 143 3588	−0.355 756 506 41	−0.059 629 327 371
7	26.668 661 9960	+0.319 169 053 10	+0.044 003 938 259
8	30.668 323 3409	−0.290 735 829 17	−0.033 813 713 976
9	34.668 073 8224	+0.267 891 182 61	+0.026 799 931 621
10	38.667 883 3469	−0.249 062 532 82	−0.021 766 340 75
n	$D_n R_n'(1)$	$H_n R_n'(1)$	
1	−1.497 549 110 17	−1.372 765 229 89	
2	−1.087 655 912 42	−0.680 054 478 86	
3	−0.925 722 120 30	−0.365 061 205 16	
4	−0.830 836 907 05	−0.231 099 090 47	
5	−0.765 838 376 14	−0.161 444 882 45	
6	−0.717 371 131 79	−0.120 240 550 18	
7	−0.679 244 328 13	−0.093 647 630 26	
8	−0.648 124 422 47	−0.075 379 405 09	
9	−0.622 028 147 07	−0.062 227 922 71	
10	−0.599 688 075 36	−0.052 408 585 29	

The first 10 of them, calculated to 12 significant digits (with the aid of the *FindRoot* library program of Mathematica®)), are listed in table 2. Due to the choice of the minus sign in equations (41), all the series $S_i(\zeta)$ are strictly positive (see table 2). They are also convergent for any $\zeta > 0$. The larger the value of ζ, the faster the convergence. For $\zeta \to \infty$, one has $S_i(\zeta) \to 0$, the leading order term being obviously the first term of the respective series. Thus, the asymptotic values of $\tilde{q}_w(\zeta; Br)$, $\Delta\tilde{T}_b(\zeta; Br)$ and $Nu(\zeta; Br)$, as being also included in table 1, can be obtained easily. In the classical Graetz case one has

$$q_{\mathrm{w}}(\infty) = 0, \tag{46a}$$

$$T_{\mathrm{w}} - T_{\mathrm{b}}(\infty) = 0, \tag{46b}$$

$$Nu(\infty) = \tfrac{1}{2}\lambda_1^2 = 3.656\,793\,457\,75. \tag{46c}$$

All the asymptotic values are independent of the Brinkman number Br. For $\zeta \to 0$, on the other hand, the series S_1 and S_2 are divergent, while the other two series converge to the values

$$S_3(0) = \tfrac{1}{4} \tag{47a}$$

and

$$S_4(0) = \tfrac{5}{24} = 0.208\,333\,333\,33. \tag{47b}$$

The limiting values (47) have been obtained by evaluating (37) for the entrance temperature profile (3*a*) and comparing the result to equation (39).

The subsequent discussion of the results is based on the approximation of the sums $S_i(\zeta)$ by their first 50 terms. In this approximation, the exact values (47) are recovered with the accuracy $S_3(0) = 0.249\,677$ and $S_4(0) = 0.208\,332$, respectively. The axial distances considered will be restricted in the following to the range $\zeta \geqslant \zeta_0 = 3 \times 10^{-4}$. With this choice, the terms of the 'worst' series, series (41*a*), decrease monotonically from $\left|D_1 R_1'(1)\,\mathrm{e}^{-\lambda_1^2\zeta}\right| \leqslant 1.5$ to $\left|D_{50} R_{50}'(1)\,\mathrm{e}^{-\lambda_{50}^2\zeta}\right| \leqslant 2.502 \times 10^{-6}$.

4.2. The local Nusselt number

The main features of the local Nusselt number $Nu(\zeta; Br)$ as a function of the axial coordinate ζ are illustrated for different (positive and negative) values of the Brinkman number and for the non-uniform, and uniform entrance temperature in figures 1 and 2. The curves of figures 1(*a*) and (*b*) correspond to positive values of the Brinkman number ('fluid cooling' situation, $T_{\mathrm{w}} < T_{\mathrm{e}}$). In this case both $\tilde{q}_{\mathrm{w}}(\zeta; Br)$ and $\Delta\tilde{T}_{\mathrm{b}}(\zeta; Br)$ are negative for all $\zeta > 0$, so that the Nusselt numbers are positive. The negative $\tilde{q}_{\mathrm{w}}(\zeta; Br)$ means that heat is always transferred from the fluid to the wall, at all axial stations $\zeta > 0$. By inspection of figures 1(*a*) and (*b*), two major qualitative differences can be identified.

First of all, figure 1(*a*) shows that the Nusselt number corresponding to the non-uniform initial condition becomes independent of the value of the Brinkman number not only for $\zeta \to \infty$ (a property which holds for all initial conditions regardless the Brinkman number), but also at the downstream station $\zeta = \zeta_* = 0.002\,345\,69$ where the value of the Nusselt number coincides with its asymptotic value 9.6. This surprising phenomenon can be explained as follows. Such an intersection point can only occur if (at least) an axial position $\zeta = \zeta_*$ exists in such a way that the ratio of the corresponding expressions of $\tilde{q}_{\mathrm{w}}(\zeta; Br)$ and $\Delta\tilde{T}_{\mathrm{b}}(\zeta; Br)$, which according to table 1 is

$$\frac{\tilde{q}_{\mathrm{w}}(\zeta_*; Br)}{\Delta\tilde{T}_{\mathrm{b}}(\zeta_*; Br)} = \frac{24}{5}\,\frac{1 + \frac{1}{4Br}S_1(\zeta_*)}{1 + \frac{24}{5Br}S_3(\zeta_*)}, \tag{48}$$

becomes independent of Br. This happens obviously for that value of ζ_* which satisfies the equation

$$\tfrac{1}{4}S_1(\zeta_*) = \tfrac{24}{5}S_3(\zeta_*). \tag{49}$$

The (unique) solution of equation (49) is $\zeta_* = 0.002\,345\,69$. The corresponding value of the Nusselt number is $2\tilde{q}_{\mathrm{w}}(\zeta_*; Br)/\Delta\tilde{T}_{\mathrm{b}}(\zeta_*; Br) = 48/5 = 9.6$, as mentioned above. For

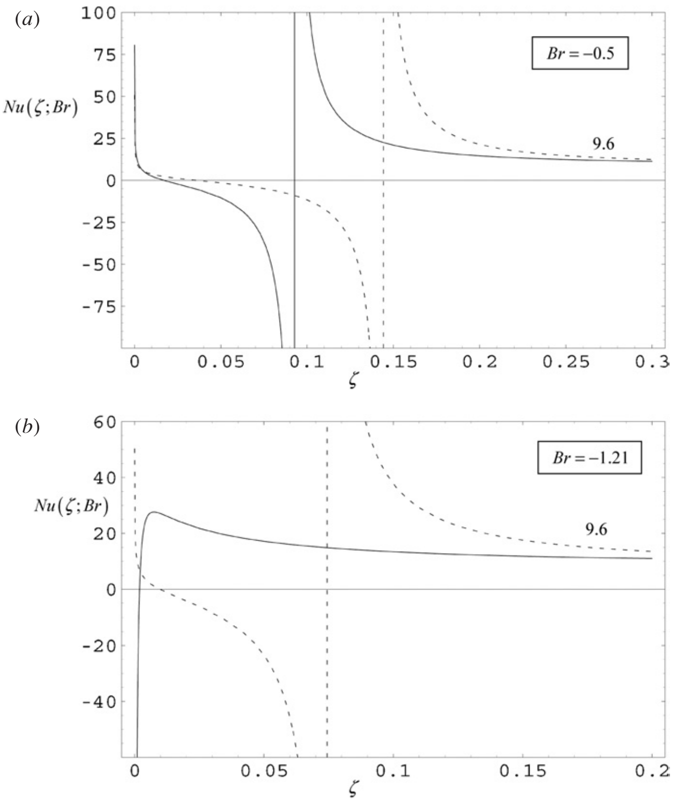

Figure 2. (*a*) Plot of the Nusselt number corresponding to the non-uniform entrance temperature (solid lines) and that corresponding to the uniform entrance temperature (dashed lines) for $Br = -0.5$. The curves emphasize a sensitive dependence of the Nusselt number on the initial conditions. (*b*) Plot of the Nusselt number corresponding to the non-uniform entrance temperature (solid lines) and that corresponding to the uniform entrance temperature (dashed lines) for $Br = -1.21$. The curves emphasize a sensitive dependence of the Nusselt number on the initial conditions.

uniform initial condition, this remarkable property of the local Nusselt number does not exist (see figure 1(*b*)).

The second major qualitative difference between the two families of curves plotted in figures 1(*a*) and (*b*) concerns the way in which they approach the (same) asymptotic value $Nu(\infty; Br) = 9.6$. The Nusselt number corresponding to the non-uniform initial condition (3*a*) approaches the asymptotic value $Nu(\infty; Br) = 9.6$ always from below (figure 1(*a*)). However, in the case of the uniform initial condition, this behaviour holds according to figure 1(*b*) only below a certain threshold value Br_{th} of the Brinkman number. These features can also be recovered analytically from the series expansion of $Nu(\zeta; Br)$ for large values of ζ. Indeed, to the leading order in ζ, this expansion of $Nu(\zeta; Br)$ reads

$$Nu(\zeta; Br) = \begin{cases} \frac{48}{5} - \frac{12}{25}\left(\frac{96}{\lambda_1^2} - 5\right) \mathrm{e}^{-\lambda_1^2\zeta} \left(-\frac{D_1}{Br}\right), & \text{non-uniform} \\ \frac{48}{5} - \frac{12}{25}\left(\frac{96}{\lambda_1^2} - 5\right) \mathrm{e}^{-\lambda_1^2\zeta} \left(-\frac{D_1}{Br} + H_1\right), & \text{uniform,} \end{cases} \tag{50}$$

equation (50) shows the mentioned properties clearly. It also allows us to determine the threshold value Br_{th} of the Brinkman number above which the Nusselt number corresponding to the uniform initial condition approaches the asymptotic value 9.6 from above. It is

$$Br_{\mathrm{th}} = \frac{D_1}{H_1} = 1.0909. \tag{51}$$

Figures 1(*a*) and (*b*) also show that for small values of ζ, with increasing $Br > 0$, the difference between the values of the Nusselt number corresponding to the uniform and non-uniform initial conditions increases rapidly.

For negative values of the Brinkman number ('fluid heating' situation, $T_{\mathrm{w}} > T_{\mathrm{e}}$) the behaviour of the Nusselt number changes substantially. This is illustrated for $Br = -0.5$ and $Br = -1.21$ in figures 2(*a*) and (*b*). One sees that for $Br < 0$, both in the cases of non-uniform and uniform entrance temperature, zeros and singularities of the Nusselt number may occur. The zeros of $Nu(\zeta; Br)$ correspond obviously to the zeros of the radial wall heat flux $\tilde{q}_{\mathrm{w}}(\zeta; Br)$ and its singularities to the zeros of the bulk temperature difference $\Delta\tilde{T}_{\mathrm{b}}(\zeta; Br)$. It can be shown that, in the case of uniform entrance temperature, $\Delta\tilde{T}_{\mathrm{b}}(\zeta; Br)$ possesses a zero for any negative value of Br, whereas for the non-uniform entrance temperature $\Delta\tilde{T}_{\mathrm{b}}(\zeta; Br)$ may vanish in the range $\zeta > 0$ only for $-6/5 \leqslant Br < 0$. The two values $Br = -0.5$ and $Br = -1.21$ have been chosen such that the latter one is out of the interval $-6/5 \leqslant Br < 0$. Accordingly, in the case of non-uniform entrance temperature, the Nusselt number distribution shown in figure 2(*b*) has no singularity for $\zeta > 0$, in contrast to that shown in figure 2(*a*).

For $\zeta \to \infty$ and $Br < 0$, all the Nu-curves approach the asymptotic value $Nu(\infty; Br) = 9.6$ always from above. This property can also be recovered analytically by a simple inspection of equation (50). Therefore, we are faced also for $Br < 0$ with a substantial dependence of the Nusselt number on the initial conditions.

The zeros of $Nu(\zeta; Br)$ existing for all negative values of the Brinkman number (i.e. in the 'fluid heating' situation, $T_{\mathrm{w}} > T_{\mathrm{e}}$) have an important physical consequence, since at the corresponding values (ζ_0, Br_0) of (ζ, Br), the radial wall heat flux $\tilde{q}_{\mathrm{w}}(\zeta; Br)$ changes sign in such a way that

$$\tilde{q}_{\mathrm{w}}(\zeta; Br_0) = \begin{cases} \geqslant 0 & \text{for} \quad 0 < \zeta \leqslant \zeta_0 \\ \leqslant 0 & \text{for} \quad \zeta \geqslant \zeta_0. \end{cases} \tag{52}$$

Hence, for a given negative value $Br = Br_0$ of the Brinkman number, the fluid is heated by the wall (i.e. $\tilde{q}_{\mathrm{w}} > 0$) only in the axial section $0 < \zeta < \zeta_0$ of the duct, while for $\zeta > \zeta_0$ the former 'fluid heating' process changes into a 'wall heating' (i.e. $\tilde{q}_{\mathrm{w}} < 0$) process. The reason is that downstream of the station $\zeta = \zeta_0$, the heat released by viscous dissipation in the bulk of the fluid overcomes the heating effect of the wall, and thus it changes the sign of $\tilde{q}_w$ from plus to minus.

4.3. The developing temperature field

The dimensionless temperature field developing in the range $z > 0$ is given by equation (30) which reduces in the three cases of interest to the following expressions:

$$\Theta(\eta, \zeta) = \sum_{n=1}^{\infty} D_n R_n(\eta)\, \mathrm{e}^{-\lambda_n^2 \zeta} \qquad \text{(classical Graetz)} \tag{53a}$$

$$\Theta(\eta, \zeta) = Br(1 - \eta^4) + \sum_{n=1}^{\infty} [D_n - Br H_n] R_n(\eta)\, \mathrm{e}^{-\lambda_n^2 \zeta} \qquad \text{(uniform entrance)} \tag{53b}$$

$$\Theta(\eta, \zeta) = Br(1 - \eta^4) + \sum_{n=1}^{\infty} D_n R_n(\eta)\, \mathrm{e}^{-\lambda_n^2 \zeta} \qquad \text{(non-uniform entrance).} \tag{53c}$$

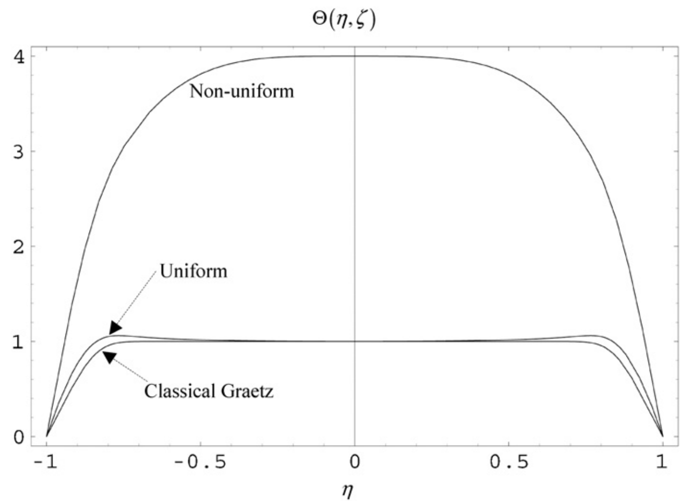

Figure 3. Plot of the dimensionless temperatures (53*b*) and (53*c*) corresponding to viscous fluid heating with uniform and non-uniform initial conditions, as well as of the dimensionless temperature (53*a*) corresponding to the classical Graetz solution, for $Br = 3$ and $\zeta = 0.001$.

Owing to equations (34), the centreline ($\eta = 0$) temperatures at the entrance station ($\zeta = 0$) can be calculated in all these cases easily. They are

$$\Theta(0, 0) = 1 \tag{54a}$$

in the classical Graetz case and for the Graetz–Brinkman problem with uniform entrance temperature, and

$$\Theta(0, 0) = 1 + Br \tag{54b}$$

for the Graetz–Brinkman problem with the non-uniform entrance temperature (3*a*), respectively. On the other hand, for $\zeta \to \infty$, i.e. far downstream from the entrance station $\zeta = 0$ where the fluid has lost any memory of its entrance state and where only the effect of the viscous dissipation is present, one obtains

$$\Theta(\eta, \zeta) \to 0 \qquad \text{(Graetz case)} \qquad (\zeta \to \infty) \tag{55a}$$

$$\Theta(\eta, \zeta) \to Br(1 - \eta^4) \qquad \text{(other two cases)} \qquad (\zeta \to \infty). \tag{55b}$$

In addition, for negligible viscous dissipation, i.e. for $Br \to 0$, as expected, cases (53*b*) and (53*c*) become coincident with (53*a*) for $\zeta \to \infty$. The largest deviations between the temperature fields (53) corresponding to uniform and non-uniform initial conditions are obtained, as expected, for large values of Br, at downstream stations close to $\zeta = 0$. This feature is illustrated in figure 3 where the developing temperature fields (53) have been plotted as functions of the dimensionless radial coordinate η for $Br = 3$ and $\zeta = 0.001$. (For the sake of intuitiveness, functions (53) have been plotted in a symmetric form by extending virtually the range of variation of η from $0 \leqslant \eta \leqslant 1$ to $-1 \leqslant \eta \leqslant 1$). Since $\zeta = 0.001$ is close to the entrance station $\zeta = 0$, equations (54) apply to the case of figure 3 with good accuracy. Since in the neighbourhood of the entrance station, the developing flow is still dominated by the initial condition, the two temperature profiles corresponding to the (same) uniform entrance temperature do occur in figure 3 as being nearly coincident (in spite of the large value of Br). The overshoots of these curves, near to $\eta = 1$, are a typical manifestation of the Gibbs phenomenon, well known from the theory of Fourier series.

5. Summary and conclusions

In the present paper, the effect of the traditional uniform entrance condition $T_{\mathrm{en}} = \mathrm{const} = T_{\mathrm{e}}$ on the thermally developing flow has been compared with that of a non-uniform entrance condition $T_{\mathrm{en}} = T_{\mathrm{en}}(r)$ assuming that the viscous dissipation is significant (Graetz–Brinkman problem). The main results and conclusions of the paper can be summarized as follows.

(1) In the presence of internal heat generation by viscous dissipation, the uniform entrance condition $T_{\mathrm{en}} = T_{\mathrm{e}}$ violates in the upstream section of the duct the thermal energy equation and thus the first principle of thermodynamics.

(2) The proper entrance condition for an *isothermal-to-isothermal* Graetz–Brinkman problem is the fully developed temperature profile $T_{\mathrm{en}}(r) = T_{\mathrm{e}} + T_* \left(1 - r^4/r_0^4\right)$ of the Poiseuille flow, obtained as the solution of energy equation with the isothermal upstream ($z \leqslant 0$) boundary condition $T|_{r=r_0} = T_{\mathrm{e}}$.

(3) The Nusselt number $Nu(\zeta; Br)$ (as the quantity of main engineering interest of the thermally developing flow) depends on the choice of the entrance condition sensitively both for positive and negative values of the Brinkman number Br, especially close to the entrance station $z = 0$ (see figure 1). The same holds for the temperature field of the thermally developing flow (see figure 3).

(4) In contrast to the uniform entrance condition, in the non-uniform case, the Nusselt number possesses for positive Br (‘fluid cooling’ situation) two remarkable properties. (1) It becomes independent of the value of the Brinkman number not only for $\zeta \to \infty$, but also at the downstream station $\zeta = \zeta_* = 0.002\,345\,69$, where the value of Nu coincides with its asymptotic value, $Nu\,(\zeta_*; Br) = Nu(\infty; Br) = 9.6$. (2) In the non-uniform case, $Nu(\zeta; Br)$ approaches its asymptotic value 9.6 always from below (see figure 1(*a*)).

(5) For all negative values of Br (i.e. in the ‘fluid heating’ situations) the radial wall heat flux $\tilde{q}_{\mathrm{w}}(\zeta; Br)$ and thus also the Nusselt number, change sign at some axial station $\zeta_0 = \zeta_0\,(Br)$. As a consequence, for a given $Br < 0$, the fluid is heated by the wall ($\tilde{q}_{\mathrm{w}} > 0$) only in the axial section $0 < \zeta < \zeta_0$ of the duct, while for $\zeta > \zeta_0$ the former ‘fluid heating’ process changes into a ‘wall heating’ ($\tilde{q}_{\mathrm{w}} < 0$) process. The reason is that the downstream of the station $\zeta = \zeta_0$, the heat released by viscous dissipation in the bulk of the fluid overcomes the heating effect of the wall. This change takes place in the case of the two entrance conditions at quite different values of $\zeta_0\,(Br)$ (see figures 2).

(6) The present consistent mathematical approach based on the non-uniform entrance temperature profile $T_{\mathrm{en}}(r) = T_{\mathrm{e}} + T_* \left(1 - r^4/r_0^4\right)$ is even simpler than the traditional one (see equation (27) and table 1).

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